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On the Convergence of Certain Finite-Difference Schemes by an Inverse-Matrix Method

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The inverse-matrix method of analyzing the convergence of the solution of a given system of finite-difference equations to the solution of the corresponding system of partial-differential equations is discussed and generalized. The convergence properties of a time- and space-centered differencing of the diffusion equation are analyzed as well as a staggered grid differencing of the Cauchy–Riemann equations. These two schemes are significant since they serve as simplified model algorithms for two recently developed methods used to calculate nonlinear aerodynamic flows.

I. INTRODUCTION

Any practical finite-difference scheme should be convergent, that is, it must have the property that as the grid is refined, the solution of the difference equation approaches the solution of the differential equation. Proving the convergence of a partial-difference scheme is generally a difficult task and one is usually satisfied with analysis of an appropriate model equation. This is especially true when practical boundary conditions are included in the analysis.

In this paper, convergence is considered from a fundamental point of view; the difference algorithm is convergent if the inverse of the matrix representing a system of linear-difference equations operates on the vector of local truncation errors such that their product approaches zero as the grid is refined. This inverse-matrix method of analyzing convergence is discussed and generalized in Section II. Previously, the approach has been applied to two-point boundary-value problems (see, e.g., [1–3]) and to the discrete analog of Laplace's equation (see, e.g., [4, 5]). For initial value problems, the inverse-matrix method is equivalent to the usual matrix method of analyzing convergence (see, e.g., [6, 7]) if the solution of the difference equations can be found by marching in a time-like direction.

Convergence proofs are provided for two recently developed difference schemes that have been successfully used in fluid flow computations but have otherwise not been analyzed. In Sections III and IV, convergence proofs are given for a time- and space-*centered* differencing of the diffusion equation subject to various boundary conditions. While this differencing is impractical for simple parabolic problems, it has application in the calculation of separated boundary-layer flow [8]. In Section V, a convergence proof is given for a staggered grid differencing of the Cauchy–Riemann equations. This difference scheme also has application in fluid mechanics. A fast direct-solution algorithm has been devised [9] which has recently been used in an iterative solution process for the nonlinear equations of subsonic and transonic aerodynamics [10].

II. Inverse-Matrix Description of Global Convergence

A system of difference equations which approximates a system of linear partial-differential equations can be written in matrix form as

$$A\mathbf{u} - \mathbf{c} = \mathbf{0}, \tag{2.1}$$

where $\mathbf{u} \in R^m$ is a real vector with m elements and A is an $m \times m$ nonsingular matrix. The elements of $\mathbf{u}$ consist of the grid function or functions for each point in the computation domain, and the vector $\mathbf{c} \in R^m$ contains specified data such as initial values, boundary values, and forcing functions. Thus, the difference operators take a system of linear partial-differential equations into a system of linear algebraic equations. It should be mentioned that in an initial value problem the time or time-like direction is also discretized with the appropriate matrix entries.

Let $\mathbf{v} \in R^m$ denote a vector whose elements consist of the exact solution to the system of partial-differential equations for each dependent variable at each grid point. Then the vector of local truncation error terms $\boldsymbol{\epsilon} \in R^m$ is defined by

$$A\mathbf{v} - \mathbf{c} = \boldsymbol{\epsilon}, \tag{2.2}$$

where $\boldsymbol{\epsilon}$ is a function of the grid spacing. If the solution to the partial-differential equation is sufficiently differentiable, each element of $\boldsymbol{\epsilon}$ can be estimated by a local Taylor series expansion. Let h be a representative spacing to which all the grid increments are referenced. Then, for a consistent difference scheme [11], $\boldsymbol{\epsilon} \to \mathbf{0}$ as $h \to 0$, but the dimension of the real vector space R^m becomes arbitrarily large.

Denote the difference $\mathbf{v} - \mathbf{u}$ by $\mathbf{e}$ and then subtract Eq. (2.1) from Eq. (2.2) to obtain

$$A\mathbf{e} = \boldsymbol{\epsilon}, \tag{2.3}$$

where it has been assumed that the difference equations and the differential equations share the same boundary and forcing functions. We can now express the error due to discretization as

$$\mathbf{e} = A^{-1}\boldsymbol{\epsilon}. \tag{2.4}$$

In general, round-off error cannot be defined at this stage since the solution process or algorithm is not specified.

A method is said to be convergent if

$$\mathbf{e} = A^{-1}\boldsymbol{\epsilon} \to \mathbf{0}, \qquad \text{as} \quad h \to 0. \tag{2.5}$$

Although $\boldsymbol{\epsilon} \to 0$ as $h \to 0$, the discretization error vector, $\mathbf{e}$, will not approach zero if A is sufficiently ill-conditioned. That is, as the grid is refined over a fixed domain of space and time, the errors may accumulate (or integrate) faster than $\boldsymbol{\epsilon}$ decays. As defined by Eq. (2.5), convergence is a global phenomenon, the difference solution must collapse upon the differential solution over the entire solution space.

The inverse-matrix method of convergence as expressed by Eq. (2.5) is not restricted to special boundary conditions or the use of just one particular class of difference formulas. In practice though, an estimate for the norm of A^{-1} can be quite difficult. However, if G is an arbitrary nonsingular matrix, the error vectors can be transformed by G, that is, $GA\mathbf{e} = G\boldsymbol{\epsilon}$, and if we define $\boldsymbol{\tau} = G\boldsymbol{\epsilon}$, the discretization error is given by

$$\mathbf{e} = (GA)^{-1}\,\boldsymbol{\tau}. \tag{2.6}$$

Eq. (2.6) is equivalent to Eq. (2.4), but it has the advantage that G may be selected to simplify the analysis, that is, the norm of $(GA)^{-1}$ may be easier to estimate than the norm of A^{-1}.

III. Convergence of a Time-Periodic Diffusion-Equation Algorithm

a. *Motivation*

The Navier–Stokes equations of fluid mechanics are often simplified by use of Prandtl's boundary-layer assumption when viscous effects are confined to thin layers in the fluid [12]. Although the boundary-layer equations are a nonlinear parabolic system of partial-differential equations, the simple model equation

$$u(\partial u/\partial t) = \partial^2 u/\partial x^2, \qquad 0 \leqslant t \leqslant t_f, \qquad 0 \leqslant x \leqslant x_f, \tag{3.1}$$

with specified initial and boundary conditions, can be used to study some of their characteristic features. In particular, for separated flow $u(x, t)$ takes on negative values in an embedded region of the field, and a finite-difference analog of Eq. (3.1) cannot be marched in time through this region.

The model Eq. (3.1) and the boundary-layer equations can be solved iteratively using combinations of backward and forward difference operators selected for $\partial u/\partial t$ according to the sign of $u(x, t)$. As an alternative to changing the difference approximation for $\partial u/\partial t$ according to the sign of u, Klineberg and Steger [8] found that accurate numerical solutions can be obtained if u_t and u_{xx} are both approximated by conventional second-order central-difference formulas. This difference analog for Eq. (3.1) forms a five-point stencil like the second-order Poisson difference operator, and the difference equations are solved simultaneously as if they were subject to Dirichlet conditions. However, since no downstream boundary condition is actually given, one-sided differencing is used at $t = t_f$ and $u(x, t_f)$ is assumed to be positive.

The use of central-difference operators in space and time which are employed in a manner appropriate to a boundary-value problem might appear to be improper for an initial value, parabolic partial-differential equation. Such a scheme must be used in conjunction with a simultaneous solution process and the differencing does not prevent data at a given time level from influencing an earlier time level. Nevertheless, convergence can be proved if the differential solution is sufficiently smooth.

Central difference analogs for parabolic problems have been used previously with overspecified Dirichlet conditions. Greenspan [13] found central difference solutions to the problem

$$u_t + f(x, t, u, u_x) = u_{xx}, \qquad 0 \leqslant x \leqslant a, \quad 0 \leqslant t \leqslant t_f, \tag{3.2a}$$

by means of the generalized Newton method, while Ban and Kuerti [14] used the same difference operators to obtain solutions to the problem

$$(x - t)\, u_t + u/2 = M^2 u_{xx}, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant 1, \tag{3.2b}$$

by successive line overrelaxation. In both problems Dirichlet conditions were imposed. Carasso and Parter [15] proved the unconditioned convergence of a central-differenced (or leap-frog) approximation to the equation

$$u_t = (a(x)\, u_x)_x + b(x)\, u_x - c(x)\, u + h(x, t), \tag{3.2c}$$

with $a(x) > 0$, and Dirichlet conditions. In another paper, Carasso [16] proved the convergence of the mildly nonlinear problem if $\Delta x = \gamma\, \Delta t$.

In this and the following section we prove convergence for the centered difference analog to the diffusion equation subject to properly specified boundary conditions;

however, we restrict our proof to the linear problem with constant coefficients. In this section we allow the coefficient of u_t to be either a positive or a negative constant. Unbounded exact solution growth for a negative coefficient is avoided by replacing the usual initial condition with a condition of temporal periodicity. In the following section we prove convergence for central differencing subject to the usual initial condition, but only a positive coefficient is allowed because our algorithm includes a change to a backward implicit differencing at $t = t_f$.

b. *Model Equation*

Consider the diffusion equation

$$\eta(\partial v/\partial t) = \partial^2 v/\partial x^2, \tag{3.3}$$

which holds on the interval $0 \leqslant t \leqslant t_f$, $0 \leqslant x \leqslant x_f$, and subject to a condition of periodicity in time

$$v(x, t) = v(x, t + t_f). \tag{3.4}$$

Periodic boundary conditions are given along $x = 0$ and $x = x_f$;

$$v(0, t) = f(t) = f(t + t_f), \tag{3.5a}$$

$$v(x_f, t) = g(t) = g(t + t_f), \tag{3.5b}$$

and because the condition of temporal periodicity is imposed, an initial condition $v(x, 0)$ cannot be specified.

A uniform grid of points is superimposed over the $x - t$ domain and labeled by $x_j = j\,\Delta x$, $t^n = n\,\Delta t$ for $j = 0, 1, 2,\ldots, J, J + 1$, and $n = 0, 1, 2,\ldots, N$, where J and N are positive integers and $t_f = N\,\Delta t$ and $x_f = (J + 1)\,\Delta x$. A difference approximation centered about (j, n)

$$\eta(u_j^{n+1} - u_j^{n-1}) - \gamma(u_{j-1}^n - 2u_j^n + u_{j+1}^n) = 0, \qquad j = 1, 2,\ldots, J, \quad n = 1, 2,\ldots, N, \tag{3.6}$$

is used to approximate Eq. (3.3) where $\gamma = 2\Delta t/(\Delta x)^2$. The local truncation error estimate of the centered difference approximation to Eq. (3.3) is

$$\epsilon_j^n = O(\Delta t^2) + O(\Delta x^2), \qquad \Delta x, \Delta t \to 0, \tag{3.7}$$

where the definition of ϵ_j^n used here is consistent with that used, for example, by Richtmeyer and Morton [11, p. 20].

The system of difference equations together with their boundary and periodicity conditions can be written as Eq. (2.1). Although the method of ordering the elements of $\mathbf{u}$ is arbitrary, we use the following "natural ordering" [17].

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ \vdots \\ \mathbf{u}^n \\ \vdots \\ \mathbf{u}^N \end{bmatrix}, \quad \text{where} \quad \mathbf{u}^n = \begin{bmatrix} u_1{}^n \\ u_2{}^n \\ \vdots \\ u_j{}^n \\ \vdots \\ u_J{}^n \end{bmatrix}.$$

The J-component subvector $\mathbf{u}^n$ has as elements the mesh function $u_j{}^n$ taken from left to right across a row on points with the index n fixed. Elements of the coefficient matrix A and the vector $\mathbf{c}$ of constants are ordered in a way which is consistent with the definition of $\mathbf{u}$. With this ordering, the matrix A is block circulant [18, 19] because of the imposed periodicity, that is, A has the form

$$A = \begin{bmatrix} T & \eta I & & \bigcirc & -\eta I \\ -\eta I & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \eta I \\ \eta I & \bigcirc & & -\eta I & T \end{bmatrix}, \qquad T = \begin{bmatrix} 2\gamma & -\gamma & & \bigcirc & \\ -\gamma & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\gamma \\ & \bigcirc & & -\gamma & 2\gamma \end{bmatrix}, \tag{3.8}$$

where $T = \gamma\, \mathrm{trid}_J(-1, 2, -1)$, T and I are order J, and A is order $N \times J$.

According to Eq. (2.5), to establish convergence for the difference algorithm we need bounds on the norm of $\mathbf{e} = 2\Delta t A^{-1}\boldsymbol{\epsilon}$ where $2\Delta t$ appears because Eq. (3.6) has been scaled by this factor. Observe that A is normal, $A^tA = AA^t$, and thus so is A^{-1}. Hence the l_2 or Euclidean induced norm of A^{-1} equals its spectral radius (see, for example, [19, 20]):

$$\| A^{-1} \|_2 = \rho(A^{-1}) = [\min | \sigma_j{}^n |]^{-1}, \tag{3.9}$$

where $\sigma_j{}^n$ denotes the distinct eigenvalues of A. Consequently,

$$\| \mathbf{e} \|_2 \leqslant 2\Delta t \rho(A^{-1}) \| \boldsymbol{\epsilon} \|_2 , \tag{3.10}$$

and an upper bound for $\| \mathbf{e} \|_2$ is readily obtained once $\rho(A^{-1})$ is determined.

An explicit formula for the eigenvalues of A can be found by using a sequence of unitary transforms to bring A into a direct sum of circulant block matrices of order J. The matrix T is symmetric so there exists an orthogonal matrix U such that [20, 21]

$$U^tTU = D, \tag{3.11}$$

where D is a diagonal matrix whose entries are the eigenvalues of T:

$$\sigma_j = 4\gamma \sin^2[j\pi/(2(J+1))], \qquad j = 1, 2,..., J. \tag{3.12}$$

Let Y be the $N \times N$ block matrix

$$Y = U \oplus U \oplus \cdots \oplus U = \begin{bmatrix} U & & & \bigcirc \\ & U & & \\ & & \ddots & \\ \bigcirc & & & U \end{bmatrix}, \tag{3.13}$$

where $\oplus$ indicates the direct sum. Then $Y^t A Y = B$ is a matrix with the same block structure as A but with blocks T replaced by D. Finally, there exists a permutation matrix P such that PBP^t is the $J \times J$ block matrix

$$PBP^t = C_1 \oplus C_2 \oplus \cdots \oplus C_j \oplus \cdots \oplus C_J, \tag{3.14}$$

where C_j is an $N \times N$ circulant matrix of the form

$$C_j = \begin{bmatrix} \sigma_j & \eta & & \bigcirc & -\eta \\ -\eta & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & \bigcirc & & \ddots & \eta \\ \eta & & & -\eta & \sigma_j \end{bmatrix}. \tag{3.15}$$

The eigenvalues of any circulant matrix are known (see [18, pp. 242–243]) and consequently the eigenvalues of A are given by

$$\sigma(A)_j^n = \sigma_j + 2i\eta \sin(2\pi n/N), \qquad j = 1, 2,..., J \quad \text{and} \quad n = 1, 2,..., N, \tag{3.16}$$

where σ_j is given by Eq. (3.12).

The spectral radius of A^{-1} is then found using the minimum eigenvalue of Eq. (3.16):

$$\begin{aligned} \min |\sigma_j^n| &= 4\gamma \sin^2[\pi/(2(J+1))] \\ &\approx 2\Delta t(\pi/x_f)^2 [1 + O(\Delta x^2)], \qquad \Delta x \to 0. \end{aligned} \tag{3.17}$$

From Eq. (3.10) it thus follows that

$$\| \mathbf{e} \|_2 \leqslant (x_f/\pi)^2 \| \boldsymbol{\epsilon} \|_2 , \tag{3.18}$$

and from the definition of the Euclidean norm

$$\| \mathbf{e} \|_2 \leqslant (x_f/\pi)^2 (JN)^{1/2} | \epsilon |_{\max} , \tag{3.19}$$

where $|\epsilon|_{\max}$ is the maximum local truncation error over the field of grid points. Finally, it follows that

$$\|\mathbf{e}\|_2 \leqslant (x_f/\pi)^2 ((x_f t_f)/(\Delta x\, \Delta t))^{1/2} [O(\Delta x^2) + O(\Delta t^2)], \tag{3.20}$$

and if, for example, $t = \alpha \Delta x$, convergence is ensured. The convergence criterion specified by Eq. (3.20) is thus quite unrestrictive and is likely conservative. Note that convergence is independent of the sign and magnitude of η.

As a practical aside, the difference equations (3.6) require simultaneous solution, and for η near zero this can be achieved by the usual successive overrelaxation procedure. If η is a constant of order one, a cyclic reduction in (n) followed by successive line overrelaxation with pentadiagonal inversion is very efficient. Fast direct solution also appears to be feasible. However, if η is a constant, Eq. (3.3) is also readily solved by a Crank–Nicolson differencing with any guessed initial condition and simply marched until the solution values found over one time period are sufficiently close to those obtained over the previous time period.

The transformations used to find the eigenvalues have frequent application in this type of analysis and they are also the basis of a class of fast direct-solution procedures; see, for example, [22] or [23]. The permutation matrix P is simply one that reorders the vector $\mathbf{u}$ from the ordering $(11, 21, 31,..., J1; 12, 22,..., J2;...; 1N,..., JN)$ to the ordering $(11, 12, 13,..., 1N; 21, 22,..., 2N;...; J1,..., JN)$. The blocks of a matrix are diagonalized by the same similarity transform if they commute (there are other special cases, see, e.g., [18, pp. 56–59]) and Y is such a similarity matrix. The columns of U are simply the eigenvectors of T which for any constant, symmetric tridiagonal matrix, $\text{trid}_J(a, b, a)$ are given by

$$\boldsymbol{\xi}_j^t = (\xi_{j1}, \xi_{j2}, ..., \xi_{jl}, ..., \xi_{jJ}), \tag{3.21}$$

with

$$\xi_{jl} = (2/(J+1))^{1/2} \sin[lj\pi/(J+1)].$$

The unitary matrix that would diagonalize C_j is comprised of the eigenvectors for any circulant matrix; see, for example, Marcus [24] or Bellman [18].

IV. Centrally Differenced Diffusion Equation with Initial Data

a. *Motivation*

In the previous section, we demonstrated that the central or leap-frog differencing of the heat equation with a time periodicity condition is convergent and is insensitive to whether the coefficient η of $\partial u/\partial t$ is positive or negative. In this section we consider the validity of this difference operator for an initial value problem where η

is assumed to be positive. Because the central-difference approximation of $\partial u/\partial t$ spans three points, it is clear that the same difference operator cannot be used at every point in the field if only one level of initial data is supplied. If in some way data are supplied at $t = \Delta t$, it is well known that explicit (or backward) use of the centered-difference operator leads to a divergent solution [11]. However, if the central-differencing scheme is applied at all but the last rows of points and if an implicit backward differencing is used at $t = t_f$, then it is shown in this section that the solution to this system of difference equations is convergent. Of course, the numerical solution must be obtained by some simultaneous solution procedure.

b. *Centrally Differenced Initial Value Problem*

Consider the diffusion equation

$$\partial u/\partial t = \kappa(\partial^2 u/\partial x^2), \qquad 0 \leqslant x \leqslant x_f, \quad 0 \leqslant t \leqslant t_f, \quad \kappa > 0, \tag{4.1}$$

subject to the initial and boundary conditions $u(x, 0) = f(x)$, $u(0, t) = g_0(t)$, and $u(x_f, t) = g_f(t)$. The usual notation for the discrete grid is introduced, and for all points $1 \leqslant n \leqslant N - 1$, $1 \leqslant j \leqslant J$, we employ the centered differencing

$$u_j^{n+1} - u_j^{n-1} - \gamma(u_{j-1}^n - 2u_j^n + u_{j+1}^n) = 0, \qquad \gamma = 2\kappa\, \Delta t/(\Delta x^2). \tag{4.2}$$

For the points $n = N$, $1 \leqslant j \leqslant J$, Eq. (4.1) is approximated by the second-order-accurate implicit backward differencing:

$$(\tfrac{1}{4})[3u_j^n - 4u_j^{n-1} + u_j^{n-2} - \gamma(u_{j-1}^n - 2u_j^n + u_{j+1}^n)] = 0, \tag{4.3}$$

where the scaling by $\frac{1}{4}$ is introduced so that u_j^{n-1} has a coefficient of unity.

The system of difference equations in the previously defined natural ordering with the assumed boundary and initial conditions generates the $N \times N$ block matrix

$$A = \begin{bmatrix} Q & I & & & \bigcirc \\ -I & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & -I & Q & I \\ \bigcirc & & & \frac{1}{4}I & -I & C \end{bmatrix}, \tag{4.4}$$

where Q is the $J \times J$ tridiagonal matrix

$$Q = \operatorname*{trid}_J\,(-\gamma, 2\gamma, -\gamma), \tag{4.5}$$

and $C = (\frac{3}{4})I + (\frac{1}{4})Q$. From the analysis of the previous section, A is found to be orthogonally similar to B;

$$B = PY^{-1}AYP^t, \tag{4.6}$$

where

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_j \oplus \cdots B_J, \tag{4.7}$$

and

$$B_j = \begin{bmatrix} \sigma_j & 1 & & & & \bigcirc \\ -1 & \cdot & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & \sigma_j & 1 \\ \bigcirc & & & \frac{1}{4} & -1 & \psi_j \end{bmatrix}. \tag{4.8}$$

Each block matrix B_j is an $N \times N$ matrix with $\psi_j = \frac{3}{4} + \sigma_j/4$, and σ_j are the eigenvalues of Q given by Eq. (3.12).

To prove convergence we shall find bounds on each element of B_j^{-1}. The matrix B_j is first split into a constant tridiagonal matrix and a perturbation matrix

$$B_j = T_j + \boldsymbol{\zeta}_N \boldsymbol{\eta}^t, \tag{4.9}$$

where

$$T_j = \underset{N}{\operatorname{trid}}(-1, \sigma_j, 1), \tag{4.10}$$

and

$$\begin{aligned} \boldsymbol{\eta}^t &= [0, 0, \ldots, 0, \tfrac{1}{4}, 0, (\tfrac{3}{4})(1 - \sigma_j)] \\ &= (\tfrac{1}{4})\, \boldsymbol{\zeta}_{N-2}^t + (\tfrac{3}{4})(1 - \sigma_j)\, \boldsymbol{\zeta}_N{}^t, \end{aligned} \tag{4.11}$$

with the unit vectors $\boldsymbol{\zeta}_{N-2}$, $\boldsymbol{\zeta}_N$ defined to be columns $N - 2$ and N of the $N \times N$ identity matrix I. The inverse of B_j can be expressed in terms of T_j^{-1} by using the Sherman–Morrison formula (see, e.g., [25, p. 122]):

$$B_j^{-1} = T_j^{-1} - \beta(T_j^{-1}\boldsymbol{\zeta}_N)[(\boldsymbol{\zeta}_{N-2}^t T_j^{-1}) + 3(1 - \sigma_j)(\boldsymbol{\zeta}_N{}^t T_j^{-1})]/4, \tag{4.12}$$

with

$$\beta = [1 + (\tfrac{1}{4})\, \tau_{N-2,N} + (\tfrac{3}{4})(1 - \sigma_j)\, \tau_{NN}]^{-1}, \tag{4.13}$$

where τ_{rs} denote the elements of T_j^{-1}, that is, $T_j^{-1} = [\tau_{rs}]$. Note that $T_j^{-1}\boldsymbol{\zeta}_N$ is simply the Nth column of T_j^{-1} and $\boldsymbol{\zeta}_N{}^t T_j^{-1}$ is the Nth row of T_j^{-1}.

Because T_j is a constant tridiagonal matrix, its inverse is known analytically.

Indeed, for a tridiagonal matrix, $T_j = \mathrm{trid}_N(a, b, c)$, the inverse elements are given by

$$\tau_{rs} = (-c)^{s-r}\, \mathscr{D}_{r-1}\mathscr{D}_{N-s}/\mathscr{D}_N\,, \qquad s \geqslant r, \tag{4.14a}$$

$$\tau_{rs} = (-a)^{r-s}\, \mathscr{D}_{N-r}\mathscr{D}_{s-1}/\mathscr{D}_N\,, \qquad s \leqslant r, \tag{4.14b}$$

where the determinant, $\mathscr{D}_l = \mathrm{Det}[\mathrm{trid}_l(a, b, c)]$, is given by the solution of the difference equation

$$\mathscr{D}_l = b\mathscr{D}_{l-1} - ac\mathscr{D}_{l-2}\,; \qquad \mathscr{D}_0 = 1, \quad \mathscr{D}_1 = b, \tag{4.15}$$

which is

$$\mathscr{D}_l = (1/\rho)[((b+\rho)/2)^{l+1} - ((b-\rho)/2)^{l+1}], \qquad \rho = (b^2 - 4ac)^{1/2}. \tag{4.16}$$

To our knowledge Eq. (4.14) is an unpublished formula derived by H. Lomax[1]; however, Fischer and Usmani [3] have published a similar formula for the trid(1, α, 1) which is diagonally similar to T_j. We also generalize a determinant relation used by Fischer and Usmani [3] to any constant tridiagonal:

$$\mathscr{D}_l = \mathscr{D}_{k+1}\mathscr{D}_{l-k-1} - ac\mathscr{D}_k\mathscr{D}_{l-k-2} \left\} \begin{array}{l} 0 \leqslant k < l, \\ \mathscr{D}_0 = 1, \quad \mathscr{D}_1 = b, \\ \mathscr{D}_m = 0, \quad m < 0. \end{array} \right. \tag{4.17}$$

In our analysis $(a, b, c) = (-1, \sigma_j, 1)$, and henceforth $\mathscr{D}_l$ will be evaluated with these particular elements. Consequently, because $\sigma_j > 0$, it follows from Eq. (4.15) with $-ac = 1$ that

$$\mathscr{D}_l > 0, \qquad \mathscr{D}_l > \mathscr{D}_{l-2}\,, \tag{4.18}$$

and if l is an even integer it follows from Eq. (4.16) that

$$\mathscr{D}_l > \mathscr{D}_{l-1}\,. \tag{4.19}$$

By restricting N to be an even integer, we can use relations (4.14)–(4.19) to establish that the absolute value of any inverse element of T_j is bounded by unity. First, consider the lower diagonal elements of T_j^{-1}. From Eq. (4.14b) every such element is positive, and combining (4.14b) with (4.17) yields

$$\tau_{rs} = \mathscr{D}_{N-r}\mathscr{D}_{s-1}/(\mathscr{D}_{k+1}\mathscr{D}_{N-k-1} + \mathscr{D}_k\mathscr{D}_{N-k-2}), \qquad s \leqslant r. \tag{4.20}$$

[1] Private communication.

Let $k = r - 1$; as k is arbitrary if $0 \leqslant k < N$, then

$$\tau_{rs} = \mathscr{D}_{N-r}\mathscr{D}_{s-1}/(\mathscr{D}_r\mathscr{D}_{N-r} + \mathscr{D}_{r-1}\mathscr{D}_{N-r-1}) \leqslant \mathscr{D}_{N-r}\mathscr{D}_{s-1}/(\mathscr{D}_r\mathscr{D}_{N-r}) = \mathscr{D}_{s-1}/\mathscr{D}_r . \quad (4.21)$$

Now if r is even, $\mathscr{D}_{s-1} < \mathscr{D}_r$ because $s \leqslant r$ and hence $\tau_{r,s} < 1$. If r is odd let $k = N - r$ and from (4.20)

$$\tau_{rs} = \mathscr{D}_{N-r}\mathscr{D}_{s-1}/(\mathscr{D}_{N-r+1}\mathscr{D}_{r-1} + \mathscr{D}_{N-r}\mathscr{D}_{r-2}) \leqslant \mathscr{D}_{N-r}\mathscr{D}_{s-1}/(\mathscr{D}_{N-r+1}\mathscr{D}_{r-1}). \quad (4.22)$$

But r is odd and N is even; therefore by (4.18) and (4.19), $\mathscr{D}_{N-r} < \mathscr{D}_{N-r+1}$ and $\mathscr{D}_{s-1} \leqslant \mathscr{D}_{r-1}$, so $\tau_{rs} < 1$. Thus, any lower diagonal element of T_j^{-1} is bounded by unity. Furthermore, because $a = -1$ and $c = 1$, it is clear from Eqs. (4.14) that $| \tau_{rs} | = | \tau_{sr} |$. Hence we have established that for any inverse element

$$| \tau_{rs} | < 1. \quad (4.23)$$

Elements of B_j^{-1} can now be bounded by using Eqs. (4.12) and (4.13). To evaluate β we use Eq. (4.14a) to evaluate $\tau_{N-2,N}$, τ_{NN} and find

$$\beta = [1 + (\tfrac{1}{4})(\mathscr{D}_{N-3}/\mathscr{D}_N) + (\tfrac{3}{4})(1 - \sigma_j)(\mathscr{D}_{N-1}/\mathscr{D}_N)]^{-1}. \quad (4.24)$$

But from (4.15), with $l = N$,

$$1 - \sigma_j = (-\mathscr{D}_N/\mathscr{D}_{N-1})(1 - (\mathscr{D}_{N-1}/\mathscr{D}_N) - (\mathscr{D}_{N-2}/\mathscr{D}_N)), \quad (4.25)$$

and consequently,

$$\beta = 4/[1 + (\mathscr{D}_{N-3}/\mathscr{D}_N) + 3(\mathscr{D}_{N-1}/\mathscr{D}_N) + 3(\mathscr{D}_{N-2}/\mathscr{D}_N)] < 4. \quad (4.26)$$

If $\sigma_j \leqslant 1$, it is evident from Eq. (4.12) that the absolute value of each element of B_j^{-1} is bounded by 5. For $\sigma_j > 1$, we need a bound on the elements of the matrix $(1 - \sigma_j)(T_j^{-1}\boldsymbol{\zeta}_N)(\boldsymbol{\zeta}_N{}^t T_j^{-1})$. Evaluating elements of this matrix using (4.25) and (4.14), we obtain

$$[(1 - \sigma_j)(T_j^{-1}\boldsymbol{\zeta}_N)(\boldsymbol{\zeta}_N{}^t T_j^{-1})]_{rs} = -(-1)^{N-r}\left(1 - \frac{\mathscr{D}_{N-2}}{\mathscr{D}_N} - \frac{\mathscr{D}_{N-1}}{\mathscr{D}_N}\right)\left(\frac{\mathscr{D}_{r-1}}{\mathscr{D}_{N-1}}\right)\left(\frac{\mathscr{D}_{s-1}}{\mathscr{D}_N}\right). \quad (4.27)$$

But if $\sigma_j \geqslant 1$, it follows directly from Eq. (4.15) that $\mathscr{D}_l > \mathscr{D}_{l-1} > 0$ and hence the absolute value of each factor of (4.27) is bounded by 1. Consequently, we find that the absolute value of each element of B_j^{-1} is bounded by 5.

With these bounds established for each element of B_j^{-1}, the convergence proof is easily completed. The norm of the discretization error vector, $\mathbf{e} = 2\Delta t\, A^{-1}\boldsymbol{\epsilon}$, is bounded by

$$\| \mathbf{e} \|_2 \leqslant 2\Delta t \, \| B^{-1} \|_2 \, \| \boldsymbol{\epsilon} \|_2 , \quad (4.28)$$

because A is orthogonally similar to B (see Eq. (4.6)). However, because B^{-1} is the direct sum of the matrices B_j^{-1};

$$\| B^{-1} \|_2 = \max_{1 \leqslant j \leqslant J} \| B_j^{-1} \|_2 \leqslant 5N, \tag{4.29}$$

where the inequality follows from the definition of the natural norm [6] and because the largest element in absolute value of the $N \times N$ matrix B_j^{-1} is $\leqslant 5$ (see, e.g., [11, p. 84]). Thus,

$$\| \mathbf{e} \|_2 \leqslant 10t_f \| \boldsymbol{\epsilon} \|_2 \leqslant 10t_f(N \cdot J)^{1/2} [O(\Delta t^2) + O(\Delta x^2)], \tag{4.30}$$

and convergence is ensured if, for example, $\Delta x = \alpha \Delta t$ where N is an even integer.

If N is an odd integer, the ratio of determinants $\mathscr{D}_{l-1}/\mathscr{D}_l$ found using (4.20) can become very large for small σ_j and we cannot provide a convergence proof by the method used in this section. The difficulty is perhaps related to the fact that the matrix $\mathrm{trid}_N(-1, 0, 1)$ is singular when N is an odd integer.

V. Staggered Grid Differencing of the Cauchy–Riemann Equations

Recently, Lomax and Martin [9] developed a fast, direct solver for the Cauchy–Riemann equations using staggered-grid difference operators. In a subsequent paper, the same authors utilized the direct solver in an iterative process for the numerical solution of the transonic small perturbation equations [10]

$$\left.\begin{aligned} u_x + v_y &= auu_x \\ u_y - v_x &= 0 \end{aligned}\right\}. \tag{5.1}$$

It appears that the staggered grid differencing of the Cauchy–Riemann equations has a number of potential applications, and consequently, in this section, we provide a formal convergence proof.

Consider the Cauchy–Riemann equations,

$$\left.\begin{aligned} u_x + v_y &= 0 \\ u_y - v_x &= 0 \end{aligned}\right\}, \tag{5.2}$$

on a rectangular domain, $0 \leqslant x \leqslant x_f$ and $0 \leqslant y \leqslant y_f$, with boundary conditions $u(0, y) = f_1(y)$, $u(x, y_f) = f_2(x)$, $v(x, 0) = f_3(x)$, and $v(x_f, y) = f_4(y)$. In a departure from the previous notation we use the same symbols u and v to denote a solution of both the differential and difference equations.

A staggered grid is introduced so that values of v are displaced from values of u by $-\Delta x/2$ and $\Delta y/2$, and the difference equations are written at different spatial locations to maintain second-order accuracy as indicated in Fig. 1. Here values of u are indicated with diamond symbols and indexed as j, k while values of v are indicated with circle symbols and indexed as $\hat{j}$, $\hat{k}$. In this notation the central difference analogs of (5.2) are

$$\begin{aligned} \mathscr{L}_1(u, v) &= (\Delta x)^{-1}(u_{j,k} - u_{j-1,k}) + (\Delta y)^{-1}(v_{\hat{j},\hat{k}} - v_{\hat{j},\hat{k}-1}) = 0, \\ \mathscr{L}_2(u, v) &= (\Delta y)^{-1}(u_{j,k+1} - u_{j,k}) - (\Delta x)^{-1}(v_{\hat{j}+1,\hat{k}} - v_{\hat{j},\hat{k}}) = 0, \end{aligned} \tag{5.3}$$

which are second-order accurate. Following Lomax and Martin, we allow the indices to range as $0 \leqslant j \leqslant J$, $1 \leqslant k \leqslant K + 1$, $1 \leqslant \hat{j} \leqslant J + 1$, and $0 \leqslant \hat{k} \leqslant K$ so that $2J \times K$ is the number of unknown net-function elements to be determined

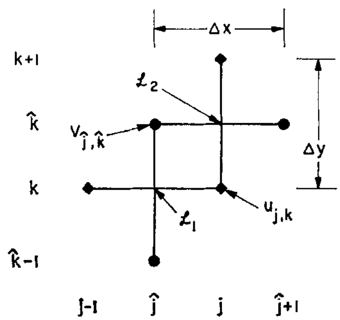

FIG. 1. Staggered computational grid for centered differencing of the Cauchy–Riemann equations.

for a given grid spacing, and $x_f = [J + (\frac{1}{2})]\,\Delta x$, $y_f = [K + (\frac{1}{2})]\,\Delta y$. A "natural" ordering of points (see Section III) is used with the set of all (j, k) or u points ordered before the set of all $(\hat{j}, \hat{k})$ or v points. Thus, the $(2JK) \times (2JK)$ matrix corresponding to Eq. (5.3) and its boundary conditions is

$$A\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = \left[\begin{array}{cccc|ccccc} L & & & & I & & & & \\ & L & & & -I & I & & & \\ & & \ddots & & & \ddots & \ddots & & \\ & & & L & & & & -I & I \\ \hline -I & I & & & L^t & & & & \\ & -I & \ddots & & & L^t & & & \\ & & \ddots & \ddots & & & \ddots & & \\ & & & I & & & & & \\ & & & -I & & & & & L^t \end{array}\right] \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_K \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_K \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix}, \tag{5.4}$$

where L is a $J \times J$ lower bidiagonal matrix

$$L = \underset{J}{\mathrm{Lbid}}\,(-\mu, \mu), \qquad \mu = \Delta y/\Delta x. \tag{5.5}$$

The error equations in terms of the major subvectors are then given by

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{e}_u \\ \mathbf{e}_v \end{bmatrix} = \Delta y \begin{bmatrix} \boldsymbol{\epsilon}_u \\ \boldsymbol{\epsilon}_v \end{bmatrix}. \tag{5.6}$$

We comment that A is an irreducible matrix [2, 4] with eigenvalues that are unknown to us; however, a matrix G is known such that GA is reduced and convergence can be proved for $\mathbf{e}_v$ uncoupled from $\mathbf{e}_u$. Indeed, we simply employ the matrix that achieves the reduction used by Lomax and Martin to decouple the u and v vectors.

Let

$$G = \begin{bmatrix} -A_{21} & A_{11} \\ 0 & I \end{bmatrix}; \tag{5.7}$$

then $GA\mathbf{e} = \Delta y \boldsymbol{\tau} \equiv \Delta y G \boldsymbol{\epsilon}$, that is,

$$\begin{bmatrix} 0 & B_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{e}_u \\ \mathbf{e}_v \end{bmatrix} = \Delta y \begin{bmatrix} -A_{21}\boldsymbol{\epsilon}_u + A_{11}\boldsymbol{\epsilon}_v \\ \boldsymbol{\epsilon}_v \end{bmatrix} = \Delta y \begin{bmatrix} \boldsymbol{\tau}_u \\ \boldsymbol{\tau}_v \end{bmatrix}, \tag{5.8}$$

where the upper, left-most block is zero because A_{21} and A_{11} commute and

$$B_{12} = -A_{21}A_{12} + A_{11}A_{22}. \tag{5.9}$$

Consequently, $\mathbf{e}_v$ is uncoupled from $\mathbf{e}_u$ and the convergence problem requires that

$$\mathbf{e}_v = \Delta y B_{12}^{-1} \boldsymbol{\tau}_u \to \mathbf{0} \tag{5.10}$$

as the grid is refined.

The matrix B_{12} has the symmetric-block tridiagonal structure

$$B_{12} = \begin{bmatrix} E & -I & & & \bigcirc \\ -I & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & E & -I \\ \bigcirc & & & -I & F \end{bmatrix}, \tag{5.11}$$

with

$$E = 2I + LL^t, \qquad F = I + LL^t. \tag{5.12}$$

The matrix LL^t is the $J \times J$ symmetric tridiagonal

$$LL^t = \mu^2 \begin{bmatrix} 1 & -1 & & & \bigcirc \\ -1 & 2 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -1 \\ \bigcirc & & & -1 & 2 \end{bmatrix}, \tag{5.13}$$

with the known eigenvalues

$$\sigma_j(LL^t) = 2\mu^2[1 + \cos(2j\pi/(2J+1))], \qquad j = 1, 2, \ldots, J. \tag{5.14}$$

Since B_{12} is symmetric, it is a normal matrix and consequently

$$\| \mathbf{e}_v \|_2 \leqslant \Delta y \| B_{12}^{-1} \|_2 \| \boldsymbol{\tau}_u \|_2 , \tag{5.15}$$

where

$$\| B_{12}^{-1} \|_2 = [\min | \sigma_{jk}(B_{12})|]^{-1}, \tag{5.16}$$

and

$$\sigma_{jk}(B_{12}) = 2\mu^2[1 + \cos(2j\pi/(2J+1))] + 2[1 + \cos(2k\pi/(2K+1))]. \tag{5.17}$$

The minimum eigenvalue occurs for $j = J$ and $k = K$, and the limiting form for vanishing mesh spacing is

$$\sigma_{JK} \sim \mu^2(\pi\, \Delta x/2x_f)^2 + (\pi\, \Delta y/2y_f)^2, \qquad \Delta x, \Delta y \to 0. \tag{5.18}$$

A bound on Eq. (5.15) is thus

$$\| \mathbf{e}_v \|_2 \leqslant \Delta y \| \boldsymbol{\tau}_u \|_2 \{(\pi/2)^2 [\mu^2(\Delta x/x_f)^2 + (\Delta y/y_f)^2]\}^{-1}. \tag{5.19}$$

A bound on $\| \boldsymbol{\tau}_u \|_2$ is now obtained from Eq. (5.8):

$$\| \boldsymbol{\tau}_u \|_2 \leqslant \| A_{21} \|_2 \| \boldsymbol{\epsilon}_u \|_2 + \| A_{11} \|_2 \| \boldsymbol{\epsilon}_v \|_2 . \tag{5.20}$$

If $\| A \|_1$ and $\| A \|_\infty$ denote the maximum absolute column sum and the maximum absolute row sum (see, e.g., [2, 5]) of a matrix, then $\| A \|_2^2 \leqslant \| A \|_1 \| A \|_\infty$, and consequently,

$$\| A_{21} \|_2^2 \leqslant \| A_{21} \|_1 \| A_{21} \|_\infty = 4, \qquad \| A_{11} \|_2^2 \leqslant \| A_{11} \|_1 \| A_{11} \|_\infty = 4\mu^2, \tag{5.21}$$

so

$$\| \boldsymbol{\tau}_u \|_2 \leqslant 2(1 + \mu) \| \boldsymbol{\epsilon}_m \|_2 , \tag{5.22}$$

where

$$\| \epsilon_m \|_2 \equiv \max(\| \epsilon_u \|_2 , \| \epsilon_v \|_2). \tag{5.23}$$

If $\epsilon_{\max}$ is the largest component of ϵ_m in absolute value, then

$$\| \epsilon_m \|_2 \leqslant (JK)^{1/2} \epsilon_{\max} . \tag{5.24}$$

Combining relations (5.19), (5.22), and (5.24), there follows

$$\| \mathbf{e}_v \|_2 \leqslant \frac{2(1+\mu)\, \Delta y (y_f x_f)^{1/2} [O(\Delta x^2) + O(\Delta y^2)]}{(\Delta x\, \Delta y)^{1/2} (\pi/2)^2 [\mu^2(\Delta x/x_f)^2 + (\Delta y/y_f)^2]} \leqslant M, \tag{5.25}$$

where M is a constant and $\Delta x = \mu\, \Delta y$ where μ is a fixed constant.

If the elements of $\mathbf{e}_v$ are all of the same order, then each element must approach zero because $\| \mathbf{e}_v \|_2^2$ remains bounded by a constant as the grid is refined. Convergence would then be proved if further study of B_{12}^{-1} should confirm that each element of $\mathbf{e}_v$ is of the same order; but a better estimate of $\| \tau_u \|_2$ is possible and is given in the following paragraph.

Assume only that the derivatives of the exact solution are continuous to one higher order than required in the truncation error analysis, then consider blocks of the term $A_{21}\epsilon_u$ of τ_u (see Eq. (5.8)):

$$A_{21}\epsilon_u = \begin{bmatrix} -I & I & & \\ & \ddots & \ddots & \\ & & -I & I \\ & & & -I \end{bmatrix} \begin{bmatrix} (\epsilon_u)_1 \\ \vdots \\ (\epsilon_u)_{K-1} \\ (\epsilon_u)_K \end{bmatrix} \approx \begin{bmatrix} \Delta y(\partial/\partial y)(\epsilon_u)_1 \\ \vdots \\ \Delta y(\partial/\partial y)(\epsilon_u)_{K-1} \\ -(\epsilon_u)_K \end{bmatrix} = \begin{bmatrix} \Delta y \tilde{\epsilon}_1 \\ \vdots \\ \Delta y \tilde{\epsilon}_{K-1} \\ -\tilde{\epsilon}_K \end{bmatrix}, \tag{5.26}$$

where the Taylor series $(\epsilon_u)_{k+1} = (\epsilon_u)_k + \Delta y(\partial \epsilon_u/\partial y)_k + \cdots$, has been used and $\tilde{\epsilon}_k$ is a vector whose elements are $O(\Delta x^2) + O(\Delta y^2)$. Except for the last block vector, A_{21} acts on ϵ_u to lower the order of the error. While previously we found

$$\| A_{21}\epsilon_1 \|_2 \leqslant 2(JK)^{1/2} \epsilon_{\max} , \tag{5.27}$$

we now obtain

$$\| A_{21}\tilde{\epsilon} \|_2 \leqslant 2J^{1/2}\tilde{\epsilon}_{\max}[1 + O(\Delta y)]^{1/2}, \tag{5.28}$$

where $\tilde{\epsilon}_{\max} = O(\Delta x^2) + O(\Delta y^2)$ but otherwise represents a term that differs from $\epsilon_{\max}$. The operator A_{11} works on ϵ_v in the same fashion and without giving details; these bounds lead to an inequality of the form

$$\| \mathbf{e}_v \|_2 \leqslant m(\Delta x)^{1/2}, \tag{5.29}$$

where as earlier $\Delta y = \mu\, \Delta x$ and where m is a constant.

To complete the analysis it must be shown that $\mathbf{e}_u \to \mathbf{0}$. The simplest approach is to simply rework the analysis using the matrix

$$G = \begin{bmatrix} I & 0 \\ A_{22} & -A_{12} \end{bmatrix}, \tag{5.30}$$

rather than the matrix G defined by Eq. (5.7). One then obtains the error relation

$$B_{21}\mathbf{e}_u = \boldsymbol{\tau}_v, \tag{5.31}$$

where

$$B_{21} = A_{22}A_{11} - A_{12}A_{21}. \tag{5.32}$$

The convergence proof for Eq. (5.31) is simply a repetition of the previous analysis since B_{21} is symmetric and has eigenvalues identical to those of B_{12}.

VI. Concluding Remarks

While the inverse-matrix method of convergence is conceptually simple, establishing bounds on the inverse of an arbitrary large matrix can be difficult. However, it is a general method that is applicable to any type of linear partial-differential equation. A major advantage of this approach is that the effects of boundary conditions which can alter convergence properties can be included in the analysis.

In this paper, the inverse-matrix method has been used to prove the convergence of a (implicit) time- and space-centered differencing of the diffusion equation as well as a staggered grid differencing of the Cauchy–Riemann equations.

References

1. R. S. Varga, "Matrix Iterative Analysis," Prentice–Hall, Englewood Cliffs, NJ, 1962.
2. J. M. Ortega, "Numerical Analysis," Academic Press, New York, 1972.
3. C. F. Fischer and R. A. Usmani, *SIAM J. Numer. Anal.* **6** (1969), 127.
4. E. L. Wachspress, "Iterative Solution of Elliptic Systems," Prentice–Hall, Englewood Cliffs, NJ, 1966.
5. E. Isaacson and H. B. Keller, "Analysis of Numerical Methods," Wiley, New York, 1966.
6. A. R. Mitchell, "Computational Methods in Partial Differential Equations," Wiley, New York, 1969.
7. G. D. Smith, "Numerical Solution of Partial Differential Equations," Oxford University Press, New York, 1965.
8. J. M. Klineberg and J. L. Steger, NASA TN D-7732, July 1974.
9. H. Lomax and E. D. Martin, *J. Computational Phys.* **15** (1974), 55.
10. E. D. Martin and H. Lomax, AIAA Paper No. 74-11, 1974.

11. R. D. Richtmyer and K. W. Morton, "Difference Methods for Initial-Value Problems" 2nd ed., Interscience, New York, 1967.
12. H. Schlichting, "Boundary Layer Theory," (4th ed.), McGraw–Hill, New York, 1960.
13. D. Greenspan, "Lectures on the Numerical Solution of Linear, Singular, and Nonlinear Differential Equations," Prentice–Hall, Englewood Cliffs, NJ, 1968.
14. S. D. Ban and G. Kuerti, *J. Fluid Mech.* **38** (1969), 109.
15. A. Carasso and S. Parter, *Math. Comp.* **24** (1970), 315.
16. A. Carasso, *Numer. Math.* **16** (1971), 304.
17. D. Young, "Iterative Solution of Large Linear Systems," Academic Press, New York, 1971.
18. R. Bellman, "Introduction to Matrix Analysis," (2nd ed.), McGraw–Hill, New York, 1970.
19. P. Lancaster, "Theory of Matrices," Academic Press, New York, 1969.
20. J. N. Franklin, "Matrix Theory," Prentice–Hall, Englewood Cliffs, NJ, 1968.
21. M. C. Pease, "Methods of Matrix Algebra," Academic Press, New York, 1965.
22. B. L. Buzbee, G. H. Golub, and C. W. Nielson, *SIAM J. Numer. Anal.* **7** (1970), 627.
23. F. Dorr, *SIAM Rev.* **12** (1970), 248.
24. M. Marcus, "Basic Theorems in Matrix Theory," *Nat. Bur. Standards Appl. Math. Ser.* **57** (1960).
25. A. S. Householder, "The Theory of Matrices in Numerical Analysis," Blaisdell, New York, 1964.